(V, σ) eukl. VR

$U = \langle g_1, \ldots, g_n \rangle \subseteq V$ Unterraum

$f \in V \qquad f^* = \sum_{i=1}^{n} \lambda_i g_i$

?

<u>Lösung</u> $\boxed{\sum_{i=1}^{n} \lambda_i \sigma(g_i, g_k) = \sigma(f, g_k) \qquad k = 1, \ldots, n}$

$b_1, \ldots, b_n \in V$ <u>Orthonormalsystem</u> $\Leftrightarrow$

① $\sigma(b_i, b_j) = 0 \quad i \neq j$

② $\sigma(b_i, b_i) = 1$

Sei $\{g_1, \ldots, g_n\}$ ONS $\Longrightarrow$

$\boxed{f^* = \sum_{i=1}^{n} \sigma(f, g_i) g_i}$ *

Satz Bez. wie oben

$P : V \longrightarrow \mathcal{U} : f \longmapsto P(f) = f^*$ beste Approx.

lineare Abbildung.

Folg Sei $V = \mathbb{R}^N$

$P : \mathbb{R}^N \longrightarrow \mathcal{U}$ ist eindeutig bestimmt durch $P(e_1), \ldots, P(e_N)$

Bem. $\bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \in \mathbb{R}^N$

$$\bar{y} = y_1 e_1 + \cdots + y_N e_N$$

$$\bar{y}^* = P(\bar{y}) = y_1 P(e_1) + \cdots + y_N P(e_N) \qquad \square$$

Fourierentwicklung

$$V = C_0(I, \mathbb{R}) = \{ f : I \to \mathbb{R} ; f \text{ stetig} \}$$

$$I = [0, 2\pi]$$

$$\sigma(f,g) = \int_0^{2\pi} f(x)\, g(x)\, dx$$

Orthonormalsystem

$$f_0(x) := \frac{1}{\sqrt{2\pi}} \qquad f_k(x) := \frac{1}{\sqrt{\pi}} \cos(kx)$$

$$g_k(x) := \frac{1}{\sqrt{\pi}} \sin(kx)$$

$$k \in \mathbb{N}$$

$$U_n := \langle f_0, f_1, \dots, f_n, g_1, \dots, g_n \rangle$$

"lineare Hülle"

Beste Approximation $f \in C_0(I, \mathbb{R})$

$$f^* = P(f) = \sigma(f, f_0) f_0 + \sum_{k=1}^{n} \sigma(f, f_k) f_k + \sum_{k=1}^{n} \sigma(f, g_k) g_k =$$

$$= \sigma(f, f_0) f_0 + \sum_{k=1}^{n} \left(\sigma(f, f_k) f_k + \sigma(f, g_k) g_k \right)$$

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \qquad f_k(x) = \frac{1}{\sqrt{\pi}} \cos(kx)$$

$$g_k(x) = \frac{1}{\sqrt{\pi}} \sin(kx)$$

$$\alpha(f, f_0) f_0 = \frac{1}{\sqrt{2\pi}} \int^{2\pi} f(x) \frac{1}{\sqrt{2\pi}} dx =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$\alpha(f, f_k) f_k = \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \cos(kx) dx \cdot \frac{1}{\sqrt{\pi}} \cos(kx)$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx \cdot \cos(kx)$$

$$\alpha(f, g_k) g_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx \cdot \sin(kx)$$

$$a_0 := \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_k := \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx$$

$$b_k := \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx.$$

$$\boxed{P(f) = f^{*} = a_0 + \sum_{k=1}^{n}\left(a_k\cos(kx) + b_k\sin(kx)\right)}$$

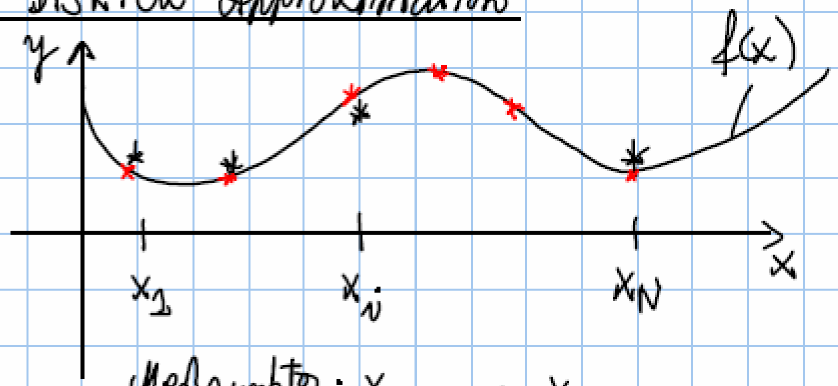
beste Approx von f in U_n

Bemerk.

$$\sum_{k=0}^{\infty}\left(a_k\cos(kx) + b_k\sin(kx)\right) = f.$$

heißt Fourierreihenentwickl. von f.

Diskrete Approximation



Meßpunkte: $x_1, \dots, x_N$

Meßwerte: $y_1, \dots, y_N$

Sei $f \in C_0(I, \mathbb{R})$ mit $I = [a, b]$

$x_i \in I \quad i = 1, \dots, N$

Auswertefunktion

$$\overline{(\;)} : C_0(I, \mathbb{R}) \longrightarrow \mathbb{R}^N$$

$$f(x) \longrightarrow \bar{f}(x) = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}$$

Lemma $\overline{(\;)} : C_0(I, \mathbb{R}) \longrightarrow \mathbb{R}^N$ ist eine lineare Abbildung

$f, g \in C_0(I, \mathbb{R})$

1) $\overline{f+g} \stackrel{?}{=} \bar{f} + \bar{g}$

2) $\overline{\lambda f} = \lambda \bar{f}$

$$\overline{f+g} = \begin{pmatrix} (f+g)(x_1) \\ \vdots \\ (f+g)(x_N) \end{pmatrix} = \begin{pmatrix} f(x_1)+g(x_1) \\ \vdots \\ f(x_N)+g(x_N) \end{pmatrix}$$

$$= \overline{f} + \overline{g}$$

$$\overline{\lambda f} = \begin{pmatrix} (\lambda f)(x_1) \\ \vdots \\ (\lambda f)(x_N) \end{pmatrix} = \begin{pmatrix} \lambda \cdot f(x_1) \\ \vdots \\ \lambda f(x_N) \end{pmatrix} =$$

$$= \lambda \cdot \overline{f} \qquad \square$$

$$\mathbb{R}^N \qquad \bar{x}_1 = \begin{pmatrix} x_{1,1} \\ \vdots \\ x_{1N} \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} x_{2,1} \\ \vdots \\ x_{2N} \end{pmatrix} \in \mathbb{R}^N$$

Eukl. Standard Skalarprodukt

$$\sigma(\bar{x}_1, \bar{x}_2) := \sum_{i=1}^{N} x_{1i} \cdot x_{2i}$$

$$\| \qquad \bar{x}_1^{\,t} \cdot \bar{x}_2$$

$$B: \beta_{1j}, \dots, \beta_{nj}$$

$$A: \alpha_{i1} \dots \alpha_{in}$$

$$c_{ij} = \sum_{k=1}^{n} \alpha_{ik} \beta_{kj}$$

$$\|\bar{x}\| = \sqrt{\sum_{i=1}^{N} x_i^{\,2}} = \sqrt{\sigma(\bar{x}, \bar{x})}$$

$$d(\bar{x}_1, \bar{x}_2) := \| \bar{x}_1 - \bar{x}_2 \|.$$

$f \in C_0(I, \mathbb{R})$ $\quad \bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$ Meßwerte-vektor

$$\| \bar{f} - \bar{y} \|$$

<u>Def.</u> <u>Beste diskrete Approximation</u>

geg. stetige $g_1, \dots, g_n \in C_0(I, \mathbb{R})$

Meßpkte: $x_1, \dots, x_N$

Meßwerte: $y_1, \dots, y_N$

$f^* \in \langle g_1, \dots, g_n \rangle$ heißt beste diskrete Approximation von $\bar{y}$ durch eine Fkt aus $\langle g_1, \dots, g_n \rangle$ wenn gilt:

$$\boxed{\forall_{g \in \langle g_1, \dots, g_n \rangle} \quad \| \bar{y} - \bar{f}^* \| \leq \| \bar{y} - \bar{g} \|}$$

<u>Bemerkung</u> $\|\bar{y}-\bar{f}^*\| \leq \|\bar{y}-\bar{g}\|$

$$\| \\ \sqrt{\sigma(\bar{y}-\bar{f}^*,\bar{y}-\bar{f}^*)} \leq \sqrt{\sigma(\bar{y}-\bar{g},\bar{y}-\bar{g})}$$

$$\Leftrightarrow \sigma(\bar{y}-\bar{f}^*,\bar{y}-\bar{f}^*) \leq \sigma(\bar{y}-\bar{g},\bar{y}-\bar{g})$$

$$\Leftrightarrow \boxed{\sum_{i=1}^{n} (y_i - f^*(x_i))^2 \leq \sum_{i=1}^{n} (y_i - g(x_i))^2}$$

$\equiv$ <u>Methode der kleinsten Quadrate</u>

von Gauss

$$\begin{array}{ccc}
C_0(I,\mathbb{R}) & \xrightarrow{\overline{(\,)}} & \mathbb{R}^N \ni \overline{y} \\
 & f \mapsto \overline{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix} & \\
\cup & = & \cup \\
\langle g_1, \ldots, g_n \rangle & \xrightarrow[\overline{(\,)}]{\text{surjektiv}} & \langle \overline{g}_1, \ldots, \overline{g}_n \rangle \ni \overline{y}^* \\
 & & =: \mathcal{U}
\end{array}$$

$$\overline{y}^* = \sum_{i=1}^{n} \lambda_i \overline{g}_i \implies$$

$$f^* := \sum_{i=1}^{n} \lambda_i g_i \quad \text{ist gesuchte Fkt.}$$

d. h. beste diskrete Approximation

Beweis $\rightarrow$ $\overline{f^*} = \sum_{i=1}^{n} \lambda_i \overline{g}_i = \overline{y}^*$ beste Approx von $\overline{y}$ in $\mathcal{U}$

$$\forall_{g \in \langle g_1, \ldots, g_n \rangle} \quad \| \overline{y} - \overline{f^*} \| = \| \overline{y} - \overline{y}^* \| \leq \| \overline{y} - \overline{g} \|$$

Beispiele

1) $g_1(x) = 1 \qquad n = 1$

$$\langle g_1(x)\rangle = \{\lambda g_1(x) ; \lambda \in \mathbb{R}\}$$
$$= \{\lambda ; \lambda \in \mathbb{R}\} = \mathbb{R}$$

$$\begin{array}{ccc} C_0(I,\mathbb{R}) & \longrightarrow & \mathbb{R}^N \ni \bar{Y} \\ \cup & & \cup \\ \langle 1 \rangle & \longrightarrow & \langle \underbrace{\begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix}}_{=\bar{1}} \rangle \ni \bar{Y}^* \end{array}$$

allgem.: $\sum_{i=1}^{n} \lambda_i \sigma(g_i, g_k) = \sigma(f, g_k)$

Anwendung $n = 1$

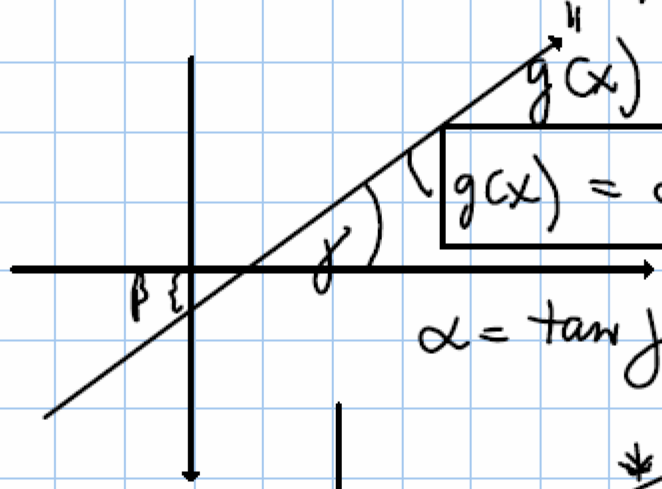
$$\lambda_1 \sigma(\bar{1}, \bar{1}) = \sigma(\bar{Y}, \bar{1})$$
$$\sigma(\bar{1}, \bar{1}) = \sum_{i=1}^{N} 1 = N, \qquad \sigma(\bar{Y}, \bar{1}) = \sum_{i=1}^{N} Y_i$$

$$\boxed{\lambda_1 = \frac{\sum_{i=1}^{N} Y_i}{N}} \longrightarrow \text{Mittelwert}$$

2) $g_1(x) = 1 \quad g_2(x) = x$

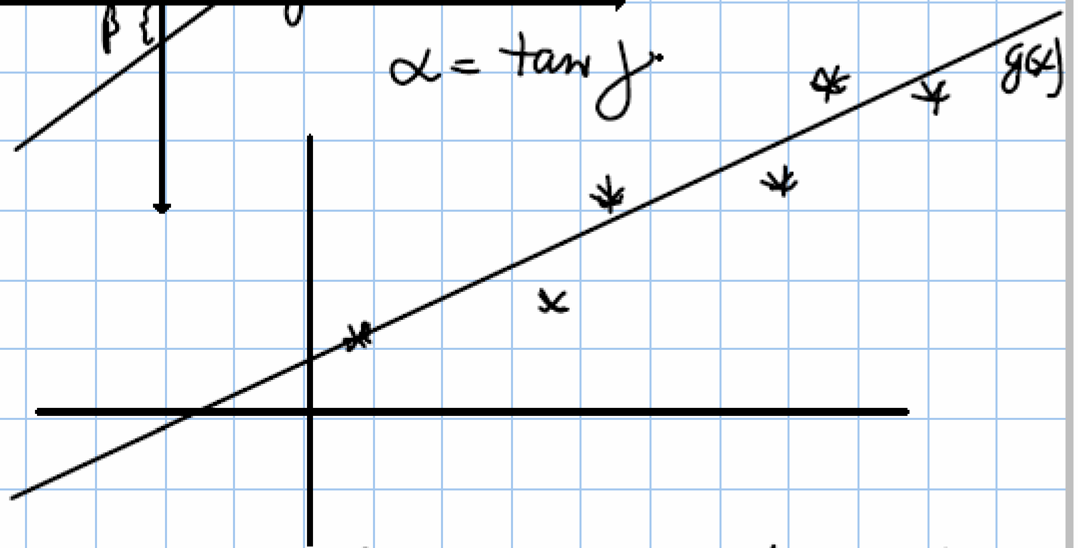
$$\langle g_1, g_2 \rangle = \{ \beta g_1 + \alpha g_2 \, ; \, \alpha, \beta \in \mathbb{R} \}$$

$$= \{ \alpha x + \beta \, ; \, \alpha, \beta \in \mathbb{R} \}$$



$$\boxed{g(x) = \alpha x + \beta}$$

$\alpha = \tan \gamma$.



beste diskrete Approximation $g(x) \equiv$

<u>lineare Regression</u>